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On the CP violation in the neutrino sector with A_4 flavour symmetry

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Abstract. *CP violation is one of the problems of the physics beyond the Standard Model. It can happen in both the quark and the lepton sectors. In the present paper, following the work arXiv:1602.07437 [hep-ph], this problem is re-considered in the lepton sector (neutrino subsector) within an extended Standard Model with an A_4 flavour discrete symmetry with a new and more convenient parametrization. As a result, a perturbative mixing matrix is derived. Then, the Dirac CP violation phase $\delta_{CP} \equiv \delta$ and the Jarlskog invariant $J_{CP} \equiv J$ are analytically obtained from theoretically derived equations leading to the solutions $\delta = \pm\frac{1}{2}\pi$. Between the two solutions, the solution $\delta = -\frac{1}{2}\pi$ (i.e., $\frac{3}{2}\pi$) is more preferable as it is more consistent with the experimental data for the inverted ordering of the neutrino masses for the gobal fit [PDG] or the normal ordering [T2K, NOνA]). A relation between δ and J is also given in terms of new parameters. The maximum value of Jarlskog invariant $|J^{max}|$ is found in the range $0.0237 < |J^{max}| < 0.034$, covering the 2022-2023 global fit values [PDG]: $|J^{max}_{PDG}| = 0.0336 \pm 0.0006$ (± 0.0019) at 1σ (3σ). Other values of J can be determined by the ralation $J(\delta)$ and approximated by Fig. 2 between two solutions.*

Keywords: neutrino mass, CP violation, Jarlskog invariant, perturbation.
Classification numbers: 13.85.-t; 03.65.Fd; 95.30.Cq; 11.30.Er and 98.80.-k.

1. Introduction

Along with other problems such as neutrino masses and mixing, the violation of the CP-symmetry, or just the CP violation (CPV), is amongst the problems beyond the explanation of the Standard Model (SM) [1–5], which so far has been the most successful model of elementary particles and their interactions, but suffers from a number of difficulties including that from the CPV. To solve these problems, a number of extensions of the SM have been suggested (for some elements of physics beyond the SM, see, for example, [6]). Here we will consider the CPV within the SM extended with an A_4 flavour symmetry based on an earlier work of some of us [7]. The CPV observed in the quark sector is not sufficient for explanation the matter-antimatter imbalance in the Universe, it is necessary to consider it in the lepton sector, including the neutrino sub-sector.

In the SM neutrinos are massless, while the experiment has shown that they are massive (as they oscillate) [8–12], although their masses are very tiny (see, for instance, [13]). This issue, as mentioned above, requires the SM to be extended [6] but in most of the models extending the SM (called beyond-SM (BSM) models for short) the quark- or lepton mixing matrices are complex, leading to the CPV as a natural consequence of these BSM models. Therefore, in general, studying CPV's is often one of the first requirements for testing a BSM model. On the other hand, the CPV within the SM (with at least three fermion families) is not strong enough to explain the current matter-antimatter imbalance in the Universe. So, the study of the BSM physics is a natural mission.

The CPV is one of Sakharov's conditions to explain the matter-antimatter imbalance in the Universe. After the P-symmetry violation (by the weak interactions) questioned and observed [14, 15], it was believed that CP-symmetry should be conserved. That is why the discovery of the CPV (in the K meson decay) by J. Cronin, V. Fitch and collaborators in 1964 shocked the physics community [16]. It was an indirect CPV, while the direct CPV was not discovered until 1999 when the experiments KTeV (Fermilab) and NA48 (CERN) announced its discovery [17, 18]. During many years following the CPV has attracted intensive theoretical and experimental investigations [19, 20]. For 12 years, between 2001 and 2013, the CPV in decays of B- and D mesons had been reported by the BaBar experiment (SLAC), the Belle experiment (KEK) and the LHCb experiment (CERN). Thus, the CPV has been confirmed in the quark sector and is expected to be observed in the lepton sector because the observed CPV in the quark sector is too small to solve the puzzle of the dominance of matter over antimatter [19]. The T2K and NOνA experiments [21, 22] have reported the first signs of the CPV in the (flavor) neutrino oscillations, thus in the lepton sector.

As said above, one of the sources of the CPV in the lepton sector is the complex phases of the PMNS lepton (neutrino) mixing matrix (which is an analog in the lepton sector of the CKM matrix in the quark sector) [19], of which only the Dirac phase can be measured in the current neutrino oscillation experiments (the Majorana phases will be the subject of our next consideration). Therefore, the determination the Dirac CPV phase (denoted below by δ_{CP} or just δ for short) is a subject of intensive investigations in both theoretical and experimental aspects (see, for example, [7, 19–26], and references therein).

Since CPV at a sufficient value (to accommodate the matter–antimatter asymmetry) cannot be predicted by the Standard Model (SM), among other reasons, the SM must be extended. In the SM, neutrinos are massless but today we know that they oscillate and, therefore, they have masses (albeit tiny and different) and mix [9–12]. It is another reason, along with the CPV, for necessity to extend the SM. A necessary condition for a complex phase to appear in a mixing matrix is that the number of fermion generations to be at least three [27]. The SM is postulated to have three generations, no more. So far, the experiment has not yet observed the fourth and other generations beyond the three of the SM. The latter have the same structure and identical quantum numbers except for the masses.

One of the simplest extensions of the SM is to impose a flavor discrete symmetry A_4 on the SM (see, e.g., [26] for a review, or [7, 28, 29] for recent results on the SM with A_4 modular symmetry). In [7] the CPV was considered in the lepton/neutrino sector within an A_4 flavor symmetric SM. In the present work we continue the latter work to investigate this problem more precisely, based on the perturbation approach previously developed in [23]. In the next section, a very brief introduction to neutrino mass and mixing, as well as the type-I seesaw mechanism is presented. This will be applied to a theoretical A_4 symmetric standard model introduced in Sect. 3 where the main results of the present paper are derived.

2. A short review on neutrino masses and mixing

The discovery of neutrino oscillations [9–11] tells us that the (flavour) neutrinos ν_α ($\alpha = e, \mu, \tau$) are mixing between mass-eigen states ν_i with different masses m_i,

$$\nu_\alpha = \sum_{i=1}^{N} U_{\alpha i} \nu_i, \tag{1}$$

where N is the number of mass-eigen states (or mass states, for short) and U is a unitary matrix called the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) mixing matrix in the case $N = 3$ (from now on we will work with $N = 3$). Therefore, the mass states are superpositions of the flavour states,

$$\nu_i = \sum_{\alpha=e,\mu,\tau} U^*_{\alpha i} \nu_\alpha, \; i = 1, 2, 3. \tag{2}$$

There are several ways to realize the mixing matrix U but in the three-neutrino theory it is usually given in the following form (in the standard parametrization):

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{13}s_{23}c_{12}e^{i\delta} & c_{23}c_{12} - s_{13}s_{23}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - s_{13}c_{23}c_{12}e^{i\delta} & -s_{23}c_{12} - s_{13}c_{23}s_{12}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \tag{3}$$

parametrised by four parameters: three angles $\theta_{12}, \theta_{23}, \theta_{13}$ and a phase $\delta \equiv \delta_{CP}$ called the Dirac phase. Here the following notations are used: $s_{ij} = sin\theta_{ij}$ và $c_{ij} = cos\theta_{ij}$ with $i, j = 1, 2, 3$. In the case of Majorana neutrinos the mixing matrix has the form

$$U_{PMNS} = U \times P, \tag{4}$$

where

$$P = \mathrm{diag}(e^{i\alpha_{21}/2}, e^{i\alpha_{31}/2}, 1), \tag{5}$$

with α_{21} and α_{31} called the Majorana phases. The neutrinos have masses but they are very very small (according to current experimental results the upper limit of neutrino masses is under 0.1 eV scale). The smallness of the neutrino masses are explained in different ways but among the most popular ones is the so-called see-saw mechanism (see details, for example, in [30–32] and references therein). There are three types of see-saw mechanism, but we will deal in the present paper with the type-I see-saw mechanism [33–37]. This type of see-saw mechanism is illustrated by the diagram in Fig. 1. The smallness of neutrino masses can be generated via the type-I see-

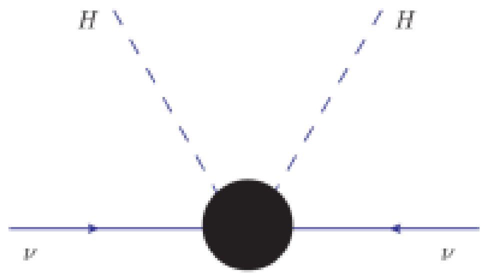

Fig. 1. Type-I see-saw neutrino effective mass [7].

saw mechanism by the introduction of a large mass scale (see more details presented, for example, in [7, 23])

$$M_\nu = -M_D(M_R)^{-1}M_D^T, \tag{6}$$

where M_ν the mass matrix of light left-handed neutrinos and M_R is the mass matrix of heavy right-handed neutrinos with masses running in the range from the eV scale until the Planck scale. We should note that the recently established upper bound $\sum m_\nu < 0.09\ eV$ of the neutrino total mass [38] allows a small, even at keV or eV, scale of M_R [39–41]. To diagonalize the matrix M_ν we use the perturbation method [42] by developing $M_\nu = M_{0\nu} + W$, where W is a small matrix compared to $M_{0\nu}$ which is the tri-bimaximal (TBM) [43] approximation of M_ν. Thus the eigenvectors $|n\rangle$ and the eigenvalues m of M_ν is developed around the eigenvectors $|n_0\rangle$ and the eigenvalues m_0 of $M_{0\nu}$,

$$|n\rangle = |n_0\rangle + \sum_{k\neq n} \lambda_{nk}|k_0\rangle \tag{7}$$

and

$$m = m_0 + \langle n_0|W|n_0\rangle, \tag{8}$$

respectively. Here

$$\lambda_{nk} = (|m_n^0| - |m_k^0|)^{-1}V_{nk}, \tag{9}$$

$$V_{nk} = \langle n_0|W|k_0\rangle. \tag{10}$$

3. A_4 flavour symmetric standard model

Let us first recall the main content of the model suggested in [7] with a focus on the see-saw-I scenario. The field ingredient of the model is given in Table 1. It consists of the scalar sector, the lepton sector and additional (sterile) neutrinos.

Table 1. Field content of an A_4 flavour symmetric model [7].

	ℓ_L	ℓ_{R_i}	Φ_h	Φ_S	$\Phi_{S'}$	$\Phi_{S''}$	N_T	N_S	$N_{S'}$	$N_{S''}$
Spin	1/2	1/2	0	0	0	0	1/2	1/2	1/2	1/2
$SU(2)_L$	2	1,1,1	2	1	1	1	1	1	1	1
A_4	3	1,1',1"	3	1	1'	1"	3	1	1'	1"

3.1. *Scalar sector*

In the scalar sector of the considered model there are four scalar fields: one SM-Higgs-like $SU(2)_L$-doublet Φ_h transforming as an A_4 triplet, and three $SU(2)_L$ singlets $\Phi_S, \Phi_{S'}$ and $\Phi_{S''}$ being also A_4 singlets. Let us see the structure of the vacuum expectation values (VEV's) of the scalar fields. The scalar field Φ_h as an A_4 triplet has the form

$$\Phi_h = (\phi_{h1}, \phi_{h2}, \phi_{h3})^T, \tag{11}$$

where ϕ_{hi} are $SU(2)_L$-doublets,

$$\phi_{hi} = \begin{pmatrix} \varphi_i^+ \\ \varphi_i^0 \end{pmatrix},\ i = 1,2,3. \tag{12}$$

The VEV structure of Φ_h is thus

$$\langle \Phi_h^0 \rangle = (\langle \varphi_1^0 \rangle, \langle \varphi_2^0 \rangle, \langle \varphi_3^0 \rangle)^T = (v_1, v_2, v_3)^T. \tag{13}$$

The VEV structure of the fields $\Phi_S, \Phi_{S'}, \Phi_{S''}$ is simply

$$\langle \Phi_S \rangle = \sigma_1, \langle \Phi_{S'} \rangle = \sigma_2, \langle \Phi_{S''} \rangle = \sigma_3. \tag{14}$$

After shifting with the VEV'S the scalar fields are represented as follows,

$$\phi_{h1} = \begin{pmatrix} \varphi_1^+ \\ v_1 + \frac{h_1 + i\eta_1}{\sqrt{2}} \end{pmatrix}, \phi_{h2} = \begin{pmatrix} \varphi_2^+ \\ v_2 + \frac{h_2 + i\eta_2}{\sqrt{2}} \end{pmatrix}, \phi_{h3} = \begin{pmatrix} \varphi_3^+ \\ v_3 + \frac{h_3 + i\eta_3}{\sqrt{2}} \end{pmatrix} \tag{15}$$

and

$$\Phi_S = \sigma_1 + \xi_1, \Phi_{S'} = \sigma_2 + \xi_2, \Phi_{S''} = \sigma_3 + \xi_3. \tag{16}$$

3.2. *Charged-lepton sector*

The lepton sector contains the SM left-handed lepton $SU(2)_L$-doublet ℓ_L now also transforming as an A_4 triplet, and the SM right-handed lepton $SU(2)_L$-singlet ℓ_{Ri} which also is an A_4-singlet. As our main goal is to get neutrino masses we consider here only the Yukawa term of the Lagrangian.

$$-L_Y = y_1(\bar{\ell}_L \Phi_h)\ell_{R1} + y_2(\bar{\ell}_L \Phi_h)''\ell_{R2} + y_3(\bar{\ell}_L \Phi_h)'\ell_{R3} + h.c. \tag{17}$$

After the symmetry breaking (the scalars to get VEV's) it becomes

$$\begin{aligned} -L_Y =& (y_1(v_1\bar{\ell}_{L1} + v_2\bar{\ell}_{L2} + v_3\bar{\ell}_{L3})\ell_{R1} + y_2(v_1\bar{\ell}_{L1} + \omega v_2\bar{\ell}_{L2} + \omega^2 v_3\bar{\ell}_{L3})\ell_{R2} \\ &+ y_3(v_1\bar{\ell}_{L1} + \omega^2 v_2\bar{\ell}_{L2} + \omega v_3\bar{\ell}_{L3})\ell_{R3}) + h.c. \end{aligned} \tag{18}$$

From here we can immediately obtain the lepton mass matrix

$$M_\ell = \begin{pmatrix} y_1 v_1 & y_2 v_1 & y_3 v_1 \\ y_1 v_2 & \omega y_2 v_2 & \omega^2 y_3 v_2 \\ y_1 v_3 & \omega^2 y_2 v_3 & \omega y_3 v_3 \end{pmatrix}. \tag{19}$$

We must work in the basis where the mass matrix of the charged leptons is diagonal. Furtunately, it can be achieved if $v_1 = v_2 = v_3 = v$ which is exactly the condition for the minimal scalar potential [7], then M_ℓ gets a diagonal form

$$M_\ell = U_L.\text{diag}(m_e, m_\mu, m_\tau), \tag{20}$$

where

$$U_L = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \tag{21}$$

with

$$m_e = \sqrt{3} y_1 v;\ m_\mu = \sqrt{3} y_2 v;\ m_\tau = \sqrt{3} y_3 v. \tag{22}$$

As $m_e \ll m_\mu \ll m_\tau$ the coefficients y_i must satisfy the constraints $y_1 \ll y_2 \ll y_3$. The neutrino part of the lepton sector will be considered separately in the next section.

4. Neutrino mixing matrix and Dirac CPV phase

As seen in [7], apart from the active neutrinos contained in the lepton electroweak doublet ℓ_L the neutrino sector contains also additional spinors treated as sterile neutrinos of which all are electroweak singlets, while one of them is an A_4 triplet, and the rest are A_4 singlet. The Yukawa term of the neutrino Lagrangian is given as follows

$$L_{Y\nu} = L^D_{Y\nu} + L^M_{Y\nu}, \tag{23}$$

where

$$L^D_{Y\nu} = y^\nu_{Ta}(\bar{\ell}_L \tilde{\Phi}_h)_{3a} N_T + y^\nu_{Tb}(\bar{\ell}_L \tilde{\Phi}_h)_{3s} N_T + y^\nu_S(\bar{\ell}_L \tilde{\Phi}_h)_1 N_S + y^\nu_{S'}(\bar{\ell}_L \tilde{\Phi}_h)_{1''} N_{S'} + y^\nu_{S''}(\bar{\ell}_L \tilde{\Phi}_h)_{1'} N_{S''} + h.c. \tag{24}$$

and

$$\begin{aligned} L^M_{Y\nu} = {} & y^\nu_{T1}(\overline{N^c_T} N_T)_1 \Phi_S + y^\nu_{T2}(\overline{N^c_T} N_T)_{1'} \Phi_{S''} + y^\nu_{T3}(\overline{N^c_T} N_T)_{1''} \Phi_{S'} \\ & + y^\nu_1(\overline{N^c_S} N_S)_1 \Phi_S + y^\nu_2(\overline{N^c_{S'}} N_{S''})_1 \Phi_S + y^\nu_3(\overline{N^c_{S'}} N_{S'})_{1''} \Phi_{S'} \\ & + y^\nu_4(\overline{N^c_S} N_{S''})_{1''} \Phi_{S'} + y^\nu_5(\overline{N^c_{S''}} N_{S''})_{1'} \Phi_{S''} + y^\nu_6(\overline{N^c_S} N_{S'})_{1'} \Phi_{S''}, \end{aligned} \tag{25}$$

where N^c denotes the charge conjugation of N. Looking at this Yukawa Lagrangian we get the Dirac mass matrix

$$M^\nu_D = \begin{pmatrix} 0 & y^\nu_{Ta} v_3 & y^\nu_{Tb} v_2 & y^\nu_S v_1 & y^\nu_{S'} v_1 & y^\nu_{S''} v_1 \\ y^\nu_{Tb} v_3 & 0 & y^\nu_{Ta} v_1 & y^\nu_S v_2 & \omega^2 y^\nu_{S'} v_2 & \omega y^\nu_{S''} v_2 \\ y^\nu_{Ta} v_2 & y^\nu_{Tb} v_1 & 0 & y^\nu_S v_3 & \omega y^\nu_{S'} v_3 & \omega^2 y^\nu_{S''} v_3 \end{pmatrix}, \tag{26}$$

where (v_1, v_2, v_3), with $v_1 = v_2 = v_3 := v$, are the VEV's of the fields $(\phi_{h1}, \phi_{h2}, \phi_{h3})$, respectively, and the Majorana mass matrix

$$M_R^\nu = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & a_{45} & a_{46} \\ 0 & 0 & 0 & a_{54} & a_{55} & a_{56} \\ 0 & 0 & 0 & a_{64} & a_{56} & a_{66} \end{pmatrix}, \tag{27}$$

where

$$\begin{aligned} a_{11} &= y^\nu_{T1}\sigma_1 + y^\nu_{T2}\sigma_2 + y^\nu_{T3}\sigma_3, \\ a_{22} &= y^\nu_{T1}\sigma_1 + \omega^2 y^\nu_{T2}\sigma_2 + \omega y^\nu_{T3}\sigma_3, \\ a_{33} &= y^\nu_{T1}\sigma_1 + \omega y^\nu_{T2}\sigma_2 + \omega^2 y^\nu_{T3}\sigma_3, \\ a_{44} &= y^\nu_1\sigma_1,\ a_{45} = a_{54} = y^\nu_6\sigma_3, \\ a_{55} &= y^\nu_3\sigma_2,\ a_{56} = a_{65} = y^\nu_2\sigma_1, \\ a_{66} &= y^\nu_5\sigma_3,\ a_{64} = a_{46} = y^\nu_4\sigma_2. \end{aligned} \tag{28}$$

From here on, we will use the notations $M_D \equiv M_D^\nu$ and $M_R \equiv M_R^\nu$ for short. The mass matrix

$$M_\nu = -M_D(M_R)^{-1}M_D^T \tag{6}$$

in the basis of the diagonalized mass matrix of the charged leptons has the form

$$\tilde{M}_\nu = U_L^\dagger M_\nu U_L^* = \begin{pmatrix} A & B & C \\ B & E & D \\ C & D & F \end{pmatrix}. \tag{29}$$

For this stage we do not need explicit expressions of the elements A, B, C, D, E and F of the latter matrix but, anyway, they are given in [7]. Unlike the latter work, where the perturbation is performed around the elements of the matrix U_{TBM}, here the argument is slightly different (but equivalent), namely, the perturbation is done around the TBM mixing angles θ_{ij}^{TBM}.

The experiment data [19] gives the mixing angles at best fit value (with 3σ CL)

$$\theta_{12} = 33.94^o,\ \theta_{23} = 46.05^o,\ \theta_{13} = 8.60^o, \tag{30}$$

corresponding to the experimental PMNS mixing matrix U_{PMNS}^{exp} at the best fit value

$$U_{PMNS}^{exp} = \begin{pmatrix} 0.8221 & 0.5695 & 0.1530e^{-i\delta} \\ -0.4337 - 0.0883e^{i\delta} & 0.6252 - 0.6024e^{i\delta} & 0.6533 \\ 0.3716 - 0.0883e^{i\delta} & -0.5373 - 0.0624e^{i\delta} & 0.7609 \end{pmatrix} \times P. \tag{31}$$

It does not deviate much from the TBM mixing matrix

$$U_{TBM} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}, \tag{32}$$

which corresponds to the mixing angles

$$\theta_{12}^{TBM} = 35.26^o,\ \theta_{23}^{TBM} = 45.00^o,\ \theta_{13}^{TBM} = 0.00^o. \tag{33}$$

We can consider the latter differing slightly from the experimentally obtained mixing angles (30)

$$\theta_{12} = \theta_{12}^{TBM} + \delta\theta_{12}, \tag{34}$$

$$\theta_{23} = \theta_{23}^{TBM} + \delta\theta_{23}, \tag{35}$$

$$\theta_{13} = \theta_{13}^{TBM} + \delta\theta_{13}, \tag{36}$$

where

$$\delta\theta_{12} = -1.32^o,\ \delta\theta_{23} = 1.05^o,\ \delta\theta_{13} = 8.60^o. \tag{37}$$

Compared with $\delta\theta_{12}$ and $\delta\theta_{23}$ the perturbation $\delta\theta_{13}$ is not really very small but it is quite small if compared with the general scale of θ_{ij}^{TBM} (e.g., $\theta_{12}^{TBM} = 35.26^o,\ \theta_{23}^{TBM} = 45.00^o$) and could serve us to obtain a good perturbative result. Therefore, we can consider U_{PMNS} perturbatively around the tri-bimaximal (TBM) mixing matrix U_{TBM} as follows

$$\tilde{U}(\theta_{12},\theta_{23},\theta_{13}) = \tilde{U}(\theta_{TBM}+\delta\theta) \approx U(\theta_{TBM}) + \Delta U(\delta\theta) \tag{38}$$

Here, we use $\tilde{U}$ for the perturbative mixing matrix instead of U used for the mixing matrix in the standard parametrization (3). The closer $\tilde{U}$ is to U, the better our model is. Thus, we can develop M_ν perturbatively around the mass $M_{0\nu}$,

$$M_\nu = M_{0\nu} + \delta M_\nu, \tag{39}$$

where $M_{0\nu}$ diagonalizable by the matrix U_{TBM} and δM is the peturbation part. Puting M_D perturbative around its TBM value (as M_R is very large we assumed it is non-perturbative)

$$M_D = M_{0D} + \delta D,\ \delta D \ll M_{0D} \text{ and } M_R = M_{0R} + \delta R,\ \delta R \approx 0$$

in (6) and using

$$M_R^{-1} = (M_R + \delta R)^{-1} \approx M_R^{-1}$$

we can write

$$M_{0\nu} = M_{0D}.(M_R)^{-1}.M_{0D}^T \tag{40}$$

and

$$\delta M_\nu = M_{0D}.M_R^{-1}.\delta D + \delta D.M_R^{-1}.M_{0D}^T. \tag{41}$$

The mass matrix $M_{0\nu}$, being a TBM limit of M_ν at $y_{S'} = y_{S"} = y_S$, has the following form

$$\begin{aligned} M_{0\nu} =& M_{0D}M_{0R}^{-1}M_{0D}^T \\ =& \begin{pmatrix} \frac{3(a_{44}+a_{46})y_S^2v^2}{(a_{44}-a_{46})(a_{44}+2a_{46})} & -\frac{3a_{46}y_S^2v^2}{a_{44}^2+a_{44}a_{46}-2a_{46}^2} & -\frac{3a_{46}y_S^2v^2}{a_{44}^2+a_{44}a_{46}-2a_{46}^2} \\ -\frac{3a_{46}y_S^2v^2}{a_{44}^2+a_{44}a_{46}-2a_{46}^2} & \frac{3a_{46}(a_{44}+a_{46})y_S^2v^2}{(a_{44}-a_{46})(a_{44}+2a_{46})(a_{44}+3a_{46})} & \frac{3(a_{44}^2+a_{44}a_{46}-a_{46}^2)y_S^2v^2}{(a_{44}-a_{46})(a_{44}+2a_{46})(a_{44}+3a_{46})} \\ -\frac{3a_{46}y_S^2v^2}{a_{44}^2+a_{44}a_{46}-2a_{46}^2} & \frac{3(a_{44}^2+2a_{44}a_{46}-a_{46}^2)y_S^2v^2}{(a_{44}-a_{46})(a_{44}+2a_{46})(a_{44}+3a_{46})} & \frac{3a_{46}(a_{44}+a_{46})y_S^2v^2}{(a_{44}-a_{46})(a_{44}+2a_{46})(a_{44}+3a_{46})} \end{pmatrix}. \end{aligned} \tag{42}$$

Then, it is not difficult to find

$$M_{0\nu}^{\text{diag}} = U_{TBM}M_{0\nu}U_{TBM}^T = \text{diag}(m_{01},m_{02},m_{03}), \tag{43}$$

with

$$m_{01} = -\frac{3y_S^2v^2}{a_{44}-a_{46}},\ m_{02} = -\frac{3y_S^2v^2}{a_{44}+2a_{46}},\ m_{03} = \frac{3y_S^2v^2}{a_{44}+3a_{46}}. \tag{44}$$

That means to obtain the true mass matrix M_ν we must make perturbation of the coupling coefficients and other parameters of the model around the TBM ones. Namely, M_ν can be obtained from $M_{0\nu}$ by perturbative shifts:

$$y_{S'} = y_S + \epsilon_1,\ y_{S"} = y_S + \epsilon_2, \tag{45}$$

with ϵ_1 and ϵ_2 ($\epsilon_1 \neq \epsilon_2$) very small compared to y_S . Then, following the method of [7] the mass matrix M_ν can be diagonalized perturbatively as follows

$$m_\nu = \tilde{U}^\dagger M_\nu \tilde{U} = \mathrm{diag}(m_1, m_2, m_3) \tag{46}$$

with the matrix $\tilde{U}$ has the form

$$\begin{aligned}\tilde{U} = & \begin{pmatrix} \sqrt{\frac{2}{3}}+\sqrt{\frac{1}{3}}x & \sqrt{\frac{1}{3}}+\sqrt{\frac{2}{3}}x & \sqrt{\frac{2}{3}}y+\sqrt{\frac{1}{3}}z \\ -\sqrt{\frac{1}{6}}+\sqrt{\frac{1}{3}}x+\sqrt{\frac{1}{2}}y^* & \sqrt{\frac{1}{3}}-\sqrt{\frac{1}{6}}x+\sqrt{\frac{1}{2}}z^* & \sqrt{\frac{1}{2}}-\sqrt{\frac{1}{6}}y+\sqrt{\frac{1}{3}}z \\ -\sqrt{\frac{1}{6}}+\sqrt{\frac{1}{3}}x-\sqrt{\frac{1}{2}}y^* & \sqrt{\frac{1}{3}}-\sqrt{\frac{1}{6}}x-\sqrt{\frac{1}{2}}z^* & -\sqrt{\frac{1}{2}}-\sqrt{\frac{1}{6}}y+\sqrt{\frac{1}{3}}z \end{pmatrix} \\ = & \ U_{TBM} + \begin{pmatrix} \sqrt{\frac{1}{3}}x & \sqrt{\frac{2}{3}}x & \sqrt{\frac{2}{3}}y+\sqrt{\frac{1}{3}}z \\ \sqrt{\frac{1}{3}}x+\sqrt{\frac{1}{2}}y^* & -\sqrt{\frac{1}{6}}x+\sqrt{\frac{1}{2}}z^* & -\sqrt{\frac{1}{6}}y+\sqrt{\frac{1}{3}}z \\ \sqrt{\frac{1}{3}}x-\sqrt{\frac{1}{2}}y^* & -\sqrt{\frac{1}{6}}x-\sqrt{\frac{1}{2}}z^* & -\sqrt{\frac{1}{6}}y+\sqrt{\frac{1}{3}}z \end{pmatrix} = U_{TBM} + \Delta U, \end{aligned} \tag{47}$$

where

$$x = -\frac{\epsilon_1+\epsilon_2}{3\sqrt{2}y_S}, \tag{48}$$

$$y = \frac{a_{46}(\epsilon_1-\epsilon_2)}{2(a_{44}+2a_{46})y_S}i, \tag{49}$$

$$z = -\frac{a_{46}(a_{44}+3a_{46})(\epsilon_1-\epsilon_2)}{\sqrt{2}(a_{44}-a_{46})(2a_{44}+5a_{46})y_S}i. \tag{50}$$

It is easy to see x, y and z (their modulus) are very small numbers. The difference with [7] is that in this paper we diagonalize M, instead of $M^\dagger M$ as done in [7] (see (82) in [7]). It is found that the matrix $\tilde{U}$ in (47) can be re-parametrized by fewer parameters. This re-parametrization allows the matrix elements of $\tilde{U}$ to be expressed in shorter forms more convenient for further development. For direct use here we only give the element U_{13} explicitly in terms of new parameters.

Comparing (47) with the matrix U_{PMNS} in the standard parametrization (3) we immediately find

$$s_{13}e^{-i\delta} = \sqrt{2/3}y + \sqrt{1/3}z. \tag{51}$$

Then, inserting y and z from (49) and (50) into (51) gives

$$s_{13}e^{-i\delta} \equiv s_{13}(\cos\delta - i\sin\delta) = -i\frac{(t-1)}{2\sqrt{6}(t+1)(t+2)}\frac{(\epsilon_1-\epsilon_2)}{y_S}. \tag{52}$$

where the new parameter $t = \frac{a_{44}}{a_{46}}$ and $\epsilon_\pm = \frac{(\epsilon_1 \pm \epsilon_2)}{y_S}$ is used. To find a solution, especially a numerical one, of this equation we need to know more about the parameters involved in. Let us demonstrate a case where we can exactly solve the equation (52).

Assuming tentatively (for simplicity) the parameters t, y_S and ϵ_i in (52) are real we readily get

$$s_{13}\cos\delta = 0 \tag{53}$$

and

$$s_{13}\sin(\delta) = \frac{(t-1)}{2\sqrt{6}(t+1)(t+2)}\epsilon_-. \tag{54}$$

As $s_{13} \neq 0$ according to the experimental data [19], the latter equations lead to

$$\cos\delta = 0 \Longrightarrow \delta = \pm\frac{\pi}{2} \tag{55}$$

and

$$s_{13} = \pm\frac{(t-1)}{2\sqrt{6}(t+1)(t+2)}\epsilon_-. \tag{56}$$

We can rewrite (56) as

$$s_{13} = \begin{cases} \frac{(t-1)}{2\sqrt{6}(t+1)(t+2)}\epsilon_- \text{ for } \delta = \frac{\pi}{2}, \\ -\frac{(t-1)}{2\sqrt{6}(t+1)(t+2)}\epsilon_- \text{ for } \delta = -\frac{\pi}{2}. \end{cases} \tag{57}$$

It is observed that s_{13} takes its TBM value $s_{13} = 0$ when $t = 1$ (i.e. $a_{44} = a_{46}$) or/and $\epsilon_1 - \epsilon_2 = 0$ (equivalent to the TBM condition $y_{S'} = y_{S''}$, as expected). Therefore, $t-1$ and $\epsilon_- \equiv \epsilon_1 - \epsilon_2$ should be small. Thus, it is more convenient to use t as a parameter along with ϵ_i and y_s. Eq. (57) allows the determination of the angle θ_{13} perturbatively and in terms of the new set of parameters.

We note that $\delta = \pm\frac{\pi}{2}$ of (55) are solutions of a mathematical equations which in principle can be applied to either the normal ordering (NO) or the inverted ordering (IO) of neutrino masses and the choice of which of the two solutions is made by fitting with the experimental data. [19, 21, 44–46]. With the reference to the experimental data the solution $\delta = -\frac{\pi}{2}$ (or $\frac{3\pi}{2}$) is more preferable in both the NO- and IO cases, for the global fit [19], NOνA [46,47] and T2K [48], all at 3σ. This result is quite good but is still conditional and, thus, may not be the actual solution because of the assumption made conditionally above about the reality of the parameters in (52), while in general some parameters can be complex. However, the method is good, we just need to further adjust the parameters. This is a subject of a later work.

The Jarlskog invariant [49], $J = \mathrm{Im}[U_{e1}U_{\mu 1}U^*_{e2}U^*_{\mu 1}]$, being a measurement of the CPV, can be expressed in terms of t as follows

$$J = \frac{(t^2 - 2t - 11)}{6\sqrt{3}(2t^3 + 7t^2 + t - 10)}\epsilon_+ \sin\ \delta = |J^{max}|\sin\delta. \tag{58}$$

The relation (58) between J and δ is depicted in Fig. 2. From the latter we see that the the

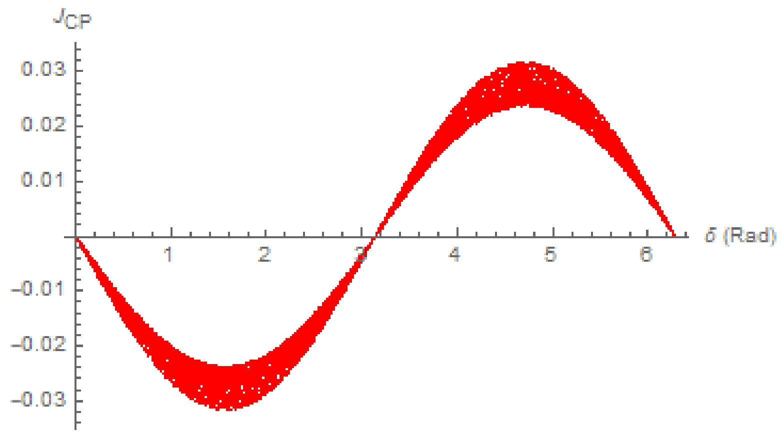

Fig. 2. The dependence between J and δ with $\epsilon_+ \in [6.10^{-3}, 10^{-2}]$ and $t \in [-2.0417, -2.028]$.

maximal value of J is within the interval $|J^{max}| \in [0.0237 - 0.0340]$, covering the latest values $|J^{max}_{PDG}| = 0.0336 \pm 0.0006$ (± 0.0019) at 1σ (3σ) by PDG. Other values of J can be detremined by Eq. (58) and approximated in Fig. 2. These values of J fit the experimental data of J if ϵ_+ and t running in the diapasons given in the capture of the figure.

5. Conclusion

The SM is a very successful theory but it also suffers some problems, such as neutrino mass and mixing, CP violation, etc., which are beyond the explanation ability of the SM. By construction neutrinos are massless in the SM, while CP violation, if any, within the SM cannot explain the matter-anti matter imbalance of the Universe. Therefore, in order to explain the problems beyond the SM the latter must be modified or extended. In the present paper we study a model extending the SM with an A_4 flavour symmetry previously proposed in [7] but studied there differently. Thus, using the type-I see-saw mechanism and following the perturbation method developed earlier [23] by some of us we obtained a theoretical mixing matrix and masses, as well as the Dirac CPV phase and Jarlskog invariant in a good agreement with the current experimental data for the CPV in the lepton sector [21, 22].

Finally, it is worth noting that one of previous works of our group [50] also considered an extended SM with A_4 flavor symmetry, but compared to the current model considered here, it was a different model with different field content. In [50], although the Dirac phase δ was found to be quite consistent with the experimental data at the time, the Jarlskog invariant was not taken into account. Moreover, in [50] the Dirac phase was determined via an analytical dependence of δ on the mixing angles θ_{ij}, while here, it is determined by comparing the perturbatively obtained U_{PMNS} matrix (of the model) with its theoretical form in the standard parametrization.

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Appendix A. Brief presentation of the group A_4

A_4 is the even permutation group of four objects [26, 51, 52]. It, geometrically isomorphic to the symmetry group of a regular tetrahedron, has 12 elements. The latter can be generated by two generators S and T satisfying the relations

$$S^2 = (ST)^3 = T^3 = 1 \tag{59}$$

The representations of this group include three one-dimensional unitary representations denoted by **1**, **1'** and **1''**, generated the generators S and T given, respectively, as follows

$$\begin{aligned} \mathbf{1} : S = 1; \qquad & T = 1, \\ \mathbf{1}' : S = 1; \qquad & T = e^{i2\pi/3} = \omega, \\ \mathbf{1}'' : S = 1; \qquad & T = e^{i4\pi/3} = \omega^2, \end{aligned}$$

and a three-dimensional unitary representation **3** generated by the generators

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \; S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \tag{60}$$

Representation theory and applications of a group often require to know the multiplication and decomposition rule of a product of its (irreducible) representations. In the case of A_4 these rules read

$$\mathbf{1} \times \mathbf{1} = \mathbf{1}, \tag{61}$$
$$\mathbf{1}' \times \mathbf{1}'' = \mathbf{1}, \tag{62}$$
$$\mathbf{1}'' \times \mathbf{1}' = \mathbf{1}, \tag{63}$$
$$\mathbf{1}' \times \mathbf{1}' = \mathbf{1}'', \tag{64}$$
$$\mathbf{1}'' \times \mathbf{1}'' = \mathbf{1}', \tag{65}$$
$$\mathbf{3} \times \mathbf{3} = \mathbf{1} + \mathbf{1}' + \mathbf{1}'' + \mathbf{3}_s + \mathbf{3}_a, \tag{66}$$

To be more explicite, the product of two triplets

$$a = (a_1, a_2, a_3), \; b = (b_1, b_2, b_3), \tag{67}$$

can be decomposed into three singlets and two triplets as follows

$$\mathbf{1} = (ab) = (a_1 b_1 + a_2 b_2 + a_3 b_3), \tag{68}$$
$$\mathbf{1}' = (ab)' = (a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3), \tag{69}$$
$$\mathbf{1}'' = (ab)'' = (a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3), \tag{70}$$
$$\mathbf{3}_a = (ab)_a = (a_2 b_3, a_3 b_1, a_1 b_2), \tag{71}$$
$$\mathbf{3}_s = (ab)_s = (a_3 b_2, a_1 b_3, a_2 b_1). \tag{72}$$