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Nonlinear multi-photon absorption of strong electromagnetic waves with electron-acoustic phonon scattering in infinite semi-parabolic plus semi-inverse squared quantum wells

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Abstract. *Theoretical study is presented on the nonlinear absorption of a strong electromagnetic wave in infinite semi-parabolic plus semi-inverse Squared Quantum Wells (ISPPSISQW) by using quantum kinetic equations for electron in the case of electron-acoustic phonon scattering. Analytical expressions for the nonlinear multi-photon absorption coefficient in the ISPPSISQW are obtained for a specific GaAs/GaAsAl in the cases of absence and presence of an external magnetic field B. The results include a second-order multi-photon process. The dependence of the nonlinear multi-photon absorption coefficient on the intensity (E_0) and the frequency (Ω) of a strong electromagnetic wave, confinement frequency of ISPPSISQW (ω_z) and the temperature is investigated. Numerical computations for a specific ISPPSISQW GaAs/GaAsAl structure show that the nonlinear multi-photon absorption coefficient decreases rapidly with increasing ω_z or Ω and increases rapidly with E_0 or T. In the case of the presence of an external magnetic field B, numerical calculations indicate the occurrence of magneto-photon-phonon resonances when the resonance conditions are met. Also, the resonance peaks tend to increase with the increase in photon energy.*

Keywords: electromagnetic waves; infinite semi-parabolic plus semi-inverse squared quantum well; multiphoton absorption; magneto-photon-phonon resonances; quantum kinetic equation.

Classification numbers: 42.50.Hz; 78.67.De; 72.10.Di.

1. Introduction

In recent years, research on the properties of low-dimensional systems has garnered significant attention from physicists and has made considerable progress. Depending on different confinement potentials, the properties of low-dimensional systems vary with different materials. Heterostructures capable of such quantum confinement are called low-dimensional semiconductor systems. Among them, special attention is paid to two-dimensional electronic systems, a model made up of a layer a few nanometers thin. This thin layer consists of a semiconductor with a small band gap, sandwiched between layers of other semiconductors with a larger band gap. The difference between the conduction band minima between the two semiconductors forms a confined quantum well, restricting the movement of electrons in a certain direction (usually the z direction). Therefore, electrons can only move freely in the x-y plane so an electronic system can be considered a two-dimensional system.

Theories about the absorption coefficient of electromagnetic waves in low-dimensional systems and bulk semiconductors have been studied by physicists for many decades [1–5]. However, multi-photon absorption in infinite semi-parabolic plus semi-inverse squared quantum wells (ISPPSISQW) has not yet been fully studied. Multi-photon absorption in ISPPSISQW with electron-optical phonon scattering has been studied in reference [6]. However, multi-photon absorption in ISPPSISQW with electron-acoustic phonon scattering at low temperatures has not yet been studied. In this paper, we investigate this case.

In this paper, we theoretically investigate the multi-photon absorption of strong electromagnetic waves in ISPPSISQW using the quantum kinetic equation method. We consider multi-photon absorption in two cases: without and with an external magnetic field. The problem is examined under the mechanism of electron-acoustic phonon scattering in the condition $|\hbar\omega_{\vec{q}} \ll \bar{\varepsilon}|$ (where $\omega_{\vec{q}}$ is the acoustic phonon frequency, $\bar{\varepsilon}$ is the average thermal kinetic energy of an electron). This paper aims to present an explicit analytical expression for the absorption coefficient in ISPPSISQW. Numerical calculations will be performed with GaAs/GaAsAl ISPPSISQW.

The paper is structured as follows: In Section 2 and Section 3, we present the model of the quantum well along with the wave function and energy spectrum of the electron, and then establish and solve the quantum kinetic equation, thereby deriving the analytical expression for the absorption coefficient of strong electromagnetic waves in both cases: with and without an external magnetic field. In Section 4, based on the obtained expression for the absorption coefficient, numerical calculations are performed to plot graphs and discuss the results. Section 5 provides the conclusion.

2. Multi-photon nonlinear absorption in the absence of an external magnetic field

2.1. Wave function and the discrete energy spectrum of the electron in ISPPSISQW

We consider an ISPPSISQW structure in which an electron moves freely in the x-y plane while being confined along the z-axis by a confinement potential of the form:

$$U(z) = \begin{cases} \infty & z < 0 \\ \frac{1}{2} m_e \omega_z^2 z^2 + \frac{\hbar\beta_z}{2mez^2} & z > 0 \end{cases}, \tag{1}$$

where m_e is the effective mass of the electron, $\hbar$ is the reduced Planck's constant, β_z and ω_z are the characteristic parameters of the potential well and the confinement frequency, respectively.

The Schrödinger equation for electrons along the z-axis has the form:

$$\left[-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial z^2}+U(z)\right]\varphi_n(z)=\varepsilon_n\varphi_n(z). \tag{2}$$

Solving this equation, we get wave functions and an energy spectrum of the electron in ISPPSISQW in the absence of an external magnetic field of the form [7, 8]:

$$\varphi_n(z)=A_n z^{2s}\exp\left(-\frac{z^2}{2\alpha_z^2}\right)L_n^{\alpha}\left(\frac{z^2}{\alpha_z^2}\right), \tag{3}$$

$$\varepsilon_n=\left(2n+1+\frac{\sqrt{1+4\beta_z}}{2}\right)\hbar\omega_z, \tag{4}$$

where $s=\frac{1}{4}(1+\sqrt{1+4\beta_z})$, $\alpha_z=\sqrt{\frac{\hbar}{m_e\omega_z}}$ and $L_n^a(x)$ is the associated Laguerre polynomial, while A_n is the wave function normalization coefficient [7, 8]:

$$A_n=\sqrt{\frac{2n!}{a_z^{1+4s}\Gamma(2s+n+1/2)}}, \tag{5}$$

where $\Gamma(x)$ represents the Gamma function.

2.2. Hamiltonian and current density expression in electron-acoustic phonon interaction

When a strong electromagnetic wave is applied to the system, with the electric field vector $\vec{E}=(0,E_0\sin\Omega t,0)$ (E_0 and Ω are the amplitude and frequency, respectively), the Hamiltonian of the electron–phonon system in ISPPSISQW can be expressed in the second quantization representation as follows [9, 10]:

$$\begin{aligned}\mathscr{H}=&\sum_{n,\vec{k}_\perp}\varepsilon_n\left(\vec{k}_\perp-\frac{e}{\hbar c}\vec{A}(t)\right)a^{\dagger}_{n,\vec{k}_\perp}a_{n,\vec{k}_\perp}\\&+\sum_{\vec{q}}\hbar\omega_{\vec{q}}b^{\dagger}_{\vec{q}}b_{\vec{q}}\\&+\sum_{n,n'}\sum_{\vec{k}_\perp,\vec{q}}C(\vec{q})I_{n,n'}(\vec{q})a^{\dagger}_{n',\vec{k}_\perp+\vec{q}_\perp}a_{n,\vec{k}_\perp}(b^{\dagger}_{-\vec{q}}+b_{\vec{q}}),\end{aligned} \tag{6}$$

where $a^{+}_{n,(\vec{k}_\perp)}$ and $a_{n,(\vec{k}_\perp)}$ ($b^{+}_{\vec{q}}$ and $b_{\vec{q}}$) are the creation and annihilation operators of electron (phonon); $\hbar\omega_{\vec{q}}$ is the phonon energy. The vector potential of laser radiation as a strong electromagnetic wave is given by:

$$A(t)=\frac{c}{\Omega}E_0\cos(\Omega t),$$

where n and n' are the band indices of states $\left|n,\vec{k}_\perp\right\rangle$ and $\left|n',\vec{k}_\perp+\vec{q}_\perp\right\rangle$, respectively. ε_n is the electron energy, $\vec{k}_\perp$ and $\vec{q}$ are the wave vectors of electrons and phonons, respectively. $C(\vec{q})$ is the electron–phonon interaction constant.

We consider the case of electron interaction with dispersion electron-acoustic phonon [9, 11]:

$$|C(\vec{q})|^2=\frac{\hbar\xi^2}{2V\rho v_s}q, \tag{7}$$

where v_s is the sound transmission speed, ρ is the crystal density, and ξ is the strain constant. The electron form factor is written as:

$$I_{n,n'}(\vec{q}) = I_{n,n'}(q_z)\delta_{\vec{k}_\perp,\vec{k}_\perp+\vec{q}_\perp} = \langle\varphi_{n'}(z)|e^{iq_z z}|\varphi_n(z)\rangle\delta_{\vec{k}_\perp,\vec{k}_\perp+\vec{q}_\perp}. \tag{8}$$

The quantum kinetic equation for electron distribution function is given by [1,9,12]:

$$i\hbar\frac{\partial f_{n,\vec{k}_\perp}(t)}{\partial t} = \langle[a^\dagger_{n,\vec{k}_\perp}a_{n,\vec{k}_\perp},H]\rangle_t, \tag{9}$$

where sub-index t denotes the statistical average value. The solution of Eq. (9) by first-order iterative approximation has the form [10,13]:

The total current density function in the case of electron-acoustic phonon scattering has the form:

$$\begin{aligned} J_\perp(t) =& \frac{e\hbar}{m_e}\sum_{n,\vec{k}_\perp}\left(\vec{k}_\perp - \frac{e}{\hbar c}\vec{A}(t)\right)f_{n,\vec{k}_\perp}(t) \\ =& -\frac{e^2E_0\eta_0\cos(\Omega t)}{m_e\Omega} + \frac{2\pi e\hbar}{m_e k\Omega}\sum_{n,n'}\sum_{\vec{k}_\perp,\vec{q}_\perp}|C(q)|^2|I_{n,n'}(\vec{q})|^2 \\ &\times\sum_{l=-\infty}^{+\infty}\vec{k}_\perp\left(J_l(J_{l+1}-J_{l-1})\right)\left(\bar{f}_{n,\vec{k}_\perp}-\bar{f}_{n',\vec{k}_\perp+\vec{q}_\perp}\right) \\ &\times(2\bar{N}_{\vec{q}}+1)\delta(\varepsilon_{n',\vec{k}_\perp+\vec{q}_\perp}-\varepsilon_{n,\vec{k}_\perp}-l\hbar\Omega)\sin(\Omega t). \end{aligned} \tag{10}$$

Here, $\eta_0 = \sum_{n,\vec{k}_\perp} f_{n,\vec{k}_\perp}$ is the electron density in ISPPSISQW, and $\delta(x)$ is the Dirac Delta function.

2.3. Multi-photon nonlinear absorption coefficient

From the total current density vector, we obtain the total absorption coefficient:

$$\alpha = \frac{16\pi^2\hbar\xi^2\Omega k_BT}{V\rho v_s^2 c\sqrt{\chi_\infty}E_0^2}\sum_{n,n'}\sum_{\vec{k}_\perp,\vec{q}_\perp}|I_{n,n'}(\vec{q})|^2(f_{n,\vec{k}_\perp}-f_{n',\vec{k}_\perp+\vec{q}_\perp})\sum_{l=-\infty}^{+\infty}lJ_l^2\left(\frac{eE_0\vec{q}_\perp}{m_e\Omega^2}\right)\delta(\varepsilon_{n'}-\varepsilon_n-l\hbar\Omega). \tag{11}$$

In the case of electron-acoustic phonon scattering, restricting the approximation $l = 1,2$ and using the expansion of the Bessel function

$$\bar{N}_{\vec{q}} = \frac{k_BT}{\hbar v_s q};\quad J_1^2(x)\approx\frac{x^2}{4}-\frac{x^4}{16};\quad J_2^2(x)\approx\frac{x^4}{64}. \tag{12}$$

The electron equilibrium distribution function is equal to the Boltzmann distribution:

$$\bar{f}_{N,n,\vec{k}_\perp} = n_0^*\exp\left(-\frac{\varepsilon_{n,\vec{k}_\perp}}{k_BT}\right), \tag{13}$$

where

$$n_0^* = \frac{2\pi\hbar^2 n_0}{Sm^*k_BT\xi},\quad \text{with } \xi = \sum_n e^{-\varepsilon_n/(k_BT)}. \tag{14}$$

We have:

$$\begin{aligned} A_{n,n',l} =& \varepsilon_{n'} - \varepsilon_n - l\hbar\Omega; \quad \Delta l = A_{n,n',l} + \frac{\hbar q_\perp^2}{2m}; \\ D_l^v =& \sum_{n,n'} \sum_{\vec{k}_\perp, \vec{q}_\perp} \left| I_{n,n'}(\vec{q}) \right|^2 \left(\bar{f}_{n,\vec{k}_\perp} - \bar{f}_{n',\vec{k}_\perp + \vec{q}_\perp} \right) \left(\frac{e\vec{E}_0 \cdot \vec{q}}{m\Omega^2} \right)^v \delta \left(\Delta l + \frac{\hbar^2 \vec{k}_\perp \cdot \vec{q}_\perp}{m} \right). \end{aligned} \tag{15}$$

Solving the equation (12) and applying the conditions (13), (14) and (15), we calculate the nonlinear absorption of strong electromagnetic waves in ISPPSISQW for the case of electron-acoustic phonon scattering with the condition $|\hbar\omega_{\vec{q}} \ll \bar{\varepsilon}|$:

$$\alpha = \frac{\pi^2 \hbar \xi^2 \Omega k_B T}{V \rho v_s^2 c \sqrt{\chi_\infty} E_0^2} \left\{ 4D_1^2 - D_1^4 + \frac{1}{2} D_2^4 \right\}, \tag{16}$$

with

$$D_1^2 = \sum_{n,n'} aG_{n,n'} \left(\frac{e\vec{E}_0}{m_e \Omega^2} \right)^2 b \left(\frac{2mA_{n,n',1}}{\hbar^2} \right)^2 K_2 \left(\frac{|A_{n,n',1}|}{2k_B T} \right), \tag{17}$$

$$D_1^4 = \sum_{n,n'} aG_{n,n'} \left(\frac{e\vec{E}_0}{m_e \Omega^2} \right)^4 b \left(\frac{2mA_{n,n',1}}{\hbar^2} \right)^3 K_3 \left(\frac{|A_{n,n',1}|}{2k_B T} \right), \tag{18}$$

$$D_2^4 = \sum_{n,n'} aG_{n,n'} \left(\frac{e\vec{E}_0}{m_e \Omega^2} \right)^4 b \left(\frac{2mA_{n,n',2}}{\hbar^2} \right)^3 K_3 \left(\frac{|A_{n,n',2}|}{2k_B T} \right). \tag{19}$$

In the above equations:

$$\begin{aligned} a =& \frac{\sqrt{2} n_0 e^{3/2}}{16\pi^2 V_0 k_B T}, \\ b =& \exp\left(-\frac{1}{k_B T} \left(\varepsilon_n + \frac{A_{n,n',l}}{2} \right) \right) - \exp\left(-\frac{1}{k_B T} \left(\varepsilon_n - \frac{A_{n,n',l}}{2} \right) \right), \\ G_{n,n'} =& \int_{-\infty}^{+\infty} \left| I_{n,n'}(q_z) \right|^2 dq_z. \end{aligned} \tag{20}$$

Here, c is the speed of light in vacuum, $K_{(v+2)/2}(x)$ is the Bessel function for the imaginary progression $(v+2)/2$, n_0 is the charge carrier density, and χ_∞ is the high-frequency dielectric coefficient.

3. Multi-photon nonlinear absorption in the case of the presence of an external magnetic field

The magnetic field runs parallel to the confinement axis Oz, denoted as $\mathbf{B} = (0,0,B)$, with a vector potential chosen in accordance with the Landau gauge, expressed as $\mathbf{A}' = (0,Bx,0)$. The wave function and its corresponding energy in the presence of an external magnetic field are then formulated as follows:

$$\Psi(r) = |N,n,k_y\rangle = \frac{1}{\sqrt{L_y}} e^{ik_y y} \phi_N(x - x_0) \phi_n(z), \tag{21}$$

$$\epsilon_{N,n} = \epsilon_N + \epsilon_n = \left(N + \frac{1}{2}\right)\hbar\omega_B + \epsilon_n. \tag{22}$$

Here, $\phi_n(z)$ and ϵ_n have been shown in Eqs. (3) and (4). $N = 0, 1, 2, \ldots$ is Landau indices, $\omega_B = \frac{eB}{m_e c}$ is cyclotron frequency and $\phi_N(x - x_0)$ is the normalized harmonic oscillator function:

$$\phi_N(x - x_0) = |N\rangle = \sqrt{\frac{1}{2^N N! \sqrt{\pi} l_B}} e^{-\frac{(x-x_0)^2}{l_B^2}} H_N\left(\frac{x - x_0}{l_B}\right), \tag{23}$$

with $H_N(x) = (-1)^N e^{x^2} \frac{d^N}{dx^N}\left[e^{-x^2}\right]$ the $N - th$ order Hermite polynomial, $x_0 = -l_B^2 k_y$ in which $l_B = \sqrt{\frac{c\hbar}{eB}}$ being the magnetic length.

Hamiltonian of the electron-acoustic phonon system in the presence of the magnetic field has the form:

$$\begin{aligned} H = & \sum_{N,n,(k_y)} \epsilon_{N,n,(k_y)} \left((k_y) - \frac{e}{\hbar c}\mathbf{A}(t)\right) a^+_{N',n',(k_y)} a_{N,n,(k_y)} \\ & + \sum_q \hbar\omega_q b_q^+ b_q \\ & + \sum_{N,N'} \sum_{n,n'} \sum_{(k_y),q} C(q) I_{n,n'}(q) J_{N,N'}(q_\perp) a^+_{N',n',(k_y)+(q_y)} a_{N,n,(k_y)} (b^+_{-q} + b_q), \end{aligned} \tag{24}$$

where $J_{N,N'}(\mathbf{q}_\perp)$ is the electron form factor under the influence of the magnetic field [14, 15]:

$$J_{N,N'}(\mathbf{q}_\perp) = \langle \varphi_{N'}(x - x_0) | e^{iq_x x} | \varphi_N(x - x_0) \rangle \delta_{\mathbf{k}_y, \mathbf{k}_y + \mathbf{q}_y}. \tag{25}$$

To determine the integral in Eq. (25), we set $u = (l_B^2 q_\perp^2)/2$ in performing calculations like those in previous works [13, 14] and obtain the electron form factor as follows [16]:

$$|J_{N,N'}(u)|^2 = \frac{N!}{N'!} e^{-u} u^{N'-N} L_N^{(N'-N)}(u). \tag{26}$$

Using a technique like the one employed to solve Eq. (9), we address the quantum kinetic equation formulated for electrons in ISPPSISQW under the influence of an external magnetic field. This approach enables us to derive the electromagnetic wave absorption coefficient.

$$\alpha = \frac{\pi^2 \hbar \xi^2 \Omega k_B T}{V \rho v_s^2 c \sqrt{\chi_\infty} E_0^2} \left\{ (4D_1^2 - D_1^4) + \frac{1}{2} D_2^4 \right\}, \tag{27}$$

with

$$D_1^2 = \sum_{N,N',n,n'} a G_{n,n'} P_2 \left(\frac{eE_0}{m_e \Omega^2}\right)^2 b\delta(A_{N,N',n,n',1}), \tag{28}$$

$$D_1^4 = \sum_{N,N',n,n'} a G_{n,n'} P_4 \left(\frac{eE_0}{m_e \Omega^2}\right)^4 b\delta(A_{N,N',n,n',1}), \tag{29}$$

$$D_2^4 = \sum_{N,N',n,n'} a G_{n,n'} P_4 \left(\frac{eE_0}{m_e \Omega^2}\right)^4 b\delta(A_{N,N',n,n',2}). \tag{30}$$

Here,

$$a = \frac{n_0 S e^{3/2}\hbar^3}{8\sqrt{\pi}\alpha_B^2 (m k_B T)^3}, \quad b = \exp\left[\frac{\varepsilon_{N,n}}{k_B T}\right] - \exp\left[\frac{\varepsilon_{N',n'}}{k_B T}\right], \quad A_{N,N',n,n',l} = \varepsilon_{N,n} - \varepsilon_{N',n'} - l\hbar\Omega, \quad (31)$$

$$P_2 = \frac{2}{l_B^2}\left(\frac{eE_0}{\sqrt{2}m\Omega^2}\right)^2 \frac{N!}{N'!}\frac{(N'-N+1)!(N'-N)!}{\Gamma(2N'-3N+3)}, \quad (32)$$

$$P_4 = \frac{4}{l_B^6}\left(\frac{eE_0}{\sqrt{2}m\Omega^2}\right)^4 \frac{N!}{N'!}\frac{(N'-N+2)!(N'-N+1)!}{\Gamma(2N'-3N+5)}, \quad (33)$$

with $S = L_x \times L_y$ the normalized area of the sample.

The Dirac delta functions in Eq. (28), Eq. (29), and Eq. (30) will diverge as their argument approaches zero. To circumvent this issue, we apply the Lorentzian transformation suggested by C. M. Van Vliet [16, 17]:

$$\delta(A_{N,N',n,n',l}) = \frac{1}{\pi}\frac{\hbar\gamma_{N,N',n,n'}}{(A_{N,N',n,n',l})^2 + \hbar^2\gamma^2_{N,N',n,n'}}, \quad (34)$$

where $\gamma_{N,N',n,n'}$ is the inverse relaxation time of the electron, given by:

$$(\gamma^{\pm}_{N,N',n,n'})^2 = \frac{1}{2}\sum_{\mathbf{q}} |C(\mathbf{q})|^2 |I_{n,n'}(\mathbf{q})|^2 |J_{N,N'}(\mathbf{q}_\perp)|^2 \left(\bar{N}_{\mathbf{q}} + \frac{1}{2} \pm \frac{1}{2}\right). \quad (35)$$

The analytical expressions for the general total absorption coefficient in the presence of a magnetic field are highly complex. In the next section, we will use numerical simulations to plot and illustrate the dependence of the total absorption coefficient on key system parameters and the external field. Additionally, we will provide a more detailed discussion on the physical significance of the analytical results obtained.

4. Numerical results and discussions

Investigate the dependence of the absorption of the strong-electromagnetic waves in infinite semi-parabolic plus semi-inverse Squared Quantum Wells (ISPPSISQW) on the external field parameters and parameters characteristic in the case of GaAs/GaAsAl ISPPSISQW. The material parameters are given as follows [18]:

Table 1. Physical parameters.

Parameter	Symbol	Value
Effective mass of electron	m	6.097×10^{-32} kg
Sound wave speed	v_s	5370 m/s
Crystal density	ρ	5320 kg/m^3
Deformation constant	ξ	13.5 eV
Boltzmann constant	k_B	1.38×10^{-23} J/K
High frequency dielectric coefficient	χ_∞	10.9
Concentration of charge carriers	η_0	10^{23} m^{-3}

4.1. Absence of an external magnetic field

4.1.1. Dependence of the absorption coefficient on the confinement frequency ω_z

Investigating the dependence of the absorption coefficient on the confinement frequency of ISPPSISQW ω_z, we obtain the graph shown in Fig. 1.

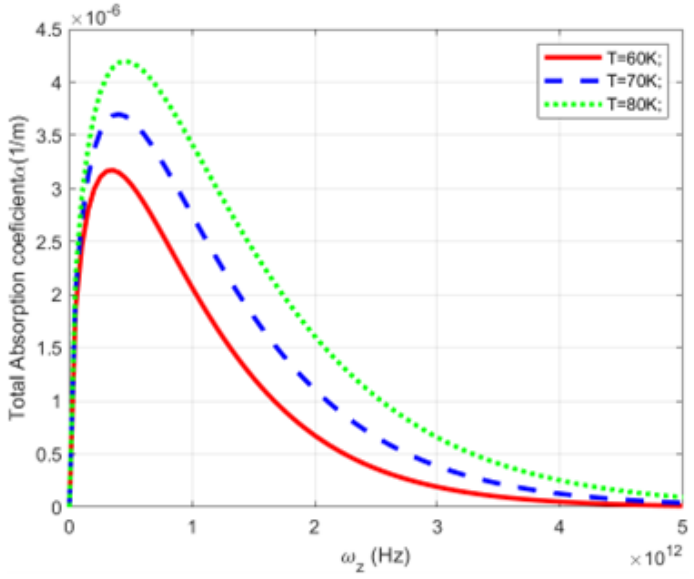


Fig. 1. Dependence of the total absorption coefficient on the confinement frequency of ISPPSISQW ω_z.

According to Fig. 1, the absorption coefficient nonlinearly depends on the confinement frequency of ISPPSISQW. Initially, the absorption coefficient increases rapidly as the confinement frequency increases and reaches its peak when the confinement frequency is around 0.25×10^{12}Hz. After that, as the confinement frequency continues to increase, the absorption coefficient decreases rapidly. Additionally, we can see that when the temperature increases, the peak shifts towards higher confinement frequencies.

The peak observed in the graph follows the resonance condition:

$$\varepsilon_{n'} - \varepsilon_n = l\hbar\Omega.$$

The resonance peak appears when $\omega_z = \Omega = 0.25 \times 10^{12}$Hz.

The results indicate that when the confinement frequency of the system increases, the ability of the strong electromagnetic wave to penetrate deeper into the material also increases. If the confinement frequency of the system continues to increase, we observe a rapid decrease in the absorption coefficient. This can be explained by the confinement frequency exceeding the resonance limit in the absorption process of strong electromagnetic waves. Additionally, we find that the asymmetric characteristics of the ISPPSISQW alter the energy levels, the electromagnetic wave absorption capability, and shift both the position of the resonance peak and the magnitude of the absorption coefficient.

4.1.2. Dependence of the absorption coefficient on the frequency of a strong electromagnetic wave Ω

Surveying the absorption coefficient on the frequency of a strong electromagnetic wave in ISPPSISQW (GaAs/GaAsAl) yields the results shown in Fig. 2.

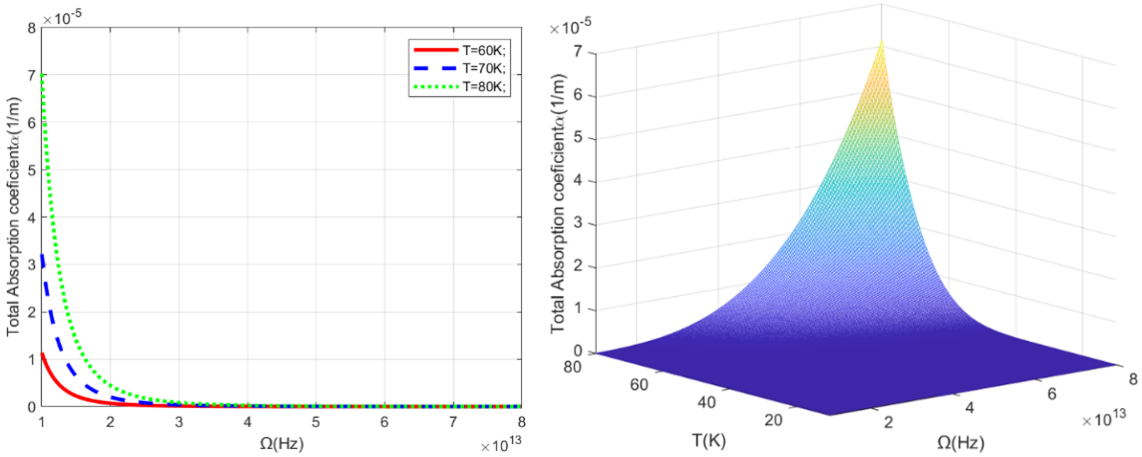


Fig. 2. (a) Dependence of the total absorption coefficient on Ω with different values of temperature T. (b) Dependence of the absorption coefficient on Ω and T.

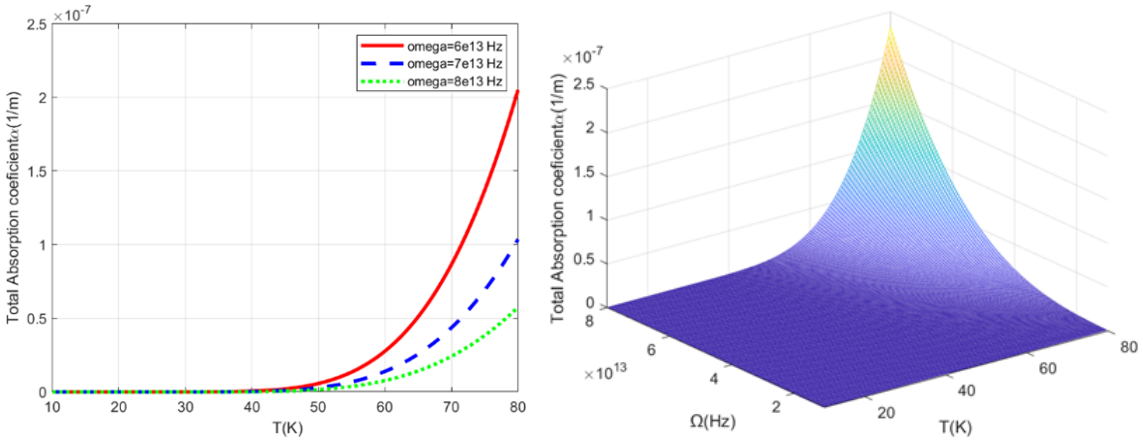
According to Fig. 2(a) and Fig. 2(b), the absorption coefficient nonlinearly depends on the frequency of strong electromagnetic waves. At a given frequency of a strong electromagnetic wave, the system's absorption coefficient is larger for higher temperatures. The absorption coefficient decreases rapidly as the frequency of a strong electromagnetic wave increases. This behavior is consistent with the energy transfer mechanism in the process of electron-acoustic phonon scattering.

At lower frequencies, the electromagnetic wave penetrates deeper into the system, leading to higher absorption by the material. As the frequency of the strong electromagnetic wave increases, the system's absorption decreases because the electron-phonon scattering becomes less effective, reducing its impact on the energy transfer process.

At higher temperatures ($T = 80K$), the absorption coefficient is greater compared to lower temperatures ($T = 70K, 60K$). This is due to the increased phonon density at higher temperatures, which enhances the scattering probability, thereby increasing the system's absorption. Additionally, acoustic phonons have low energy and are wave-vector dependent ($\vec{q}$), which causes the absorption coefficient to remain relatively small. As a result, no sharp resonance peaks are observed in the system.

4.1.3. Dependence of the absorption coefficient on the temperature T

Conducting a study on the effect of temperature on the absorption coefficient when changing the frequency of a strong electromagnetic wave, the following results were obtained (see Fig. 3).



(a) (b)

Fig. 3. (a) Influence of temperature T on the total absorption coefficient at different frequencies of a strong electromagnetic wave Ω (left). (b) Influence of temperature T and frequency of a strong electromagnetic wave Ω on the total absorption coefficient (right).

According to the graphs in Fig. 3(a) and Fig. 3(b), the total absorption coefficient strongly and nonlinearly depends on temperature. Initially, as the system's temperature increases, the absorption coefficient increases very slowly. However, when the temperature exceeds $50K$, the absorption coefficient increases rapidly and nonlinearly with the temperature rise.

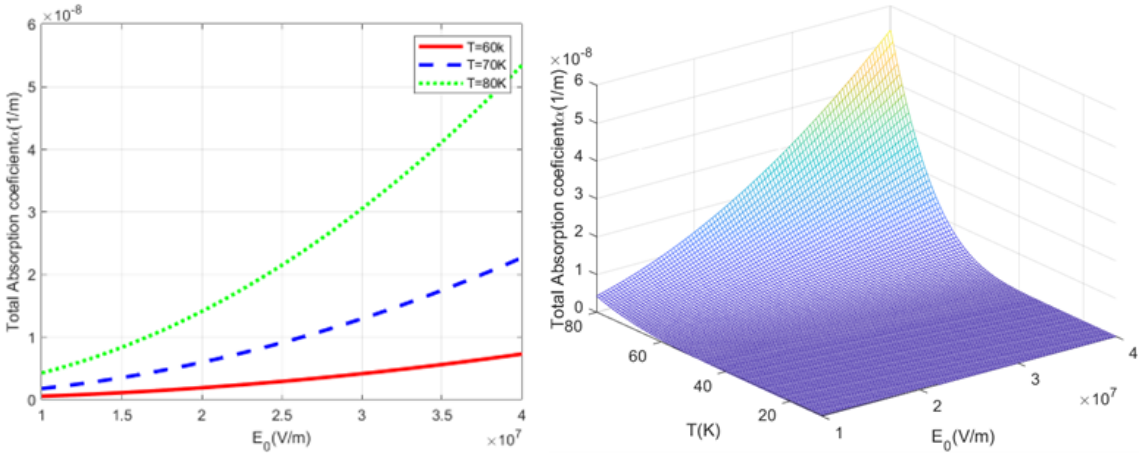
The increase in temperature leads to enhanced thermal vibrations of phonons, increasing the density of phonon states participating in the scattering process. The absorption coefficient depends on the energy exchange between electrons and phonons. The increase in temperature enhances the thermal vibrations of phonons, leading to a higher density of phonon states participating in the scattering process. The absorption coefficient depends on the probability of energy exchange between electrons and phonons.

As the system's temperature increases, it intensifies thermal vibrations and the phonon density of states during scattering. At higher frequencies of Ω (represented by the blue and green curves), the absorption coefficient tends to be lower compared to $\Omega = 6 \times 10^{13}$. However, when the frequency of a strong electromagnetic wave increases, it reduces the system's resonance capability. This demonstrates that temperature strongly affects the absorption coefficient, particularly at high frequencies of a strong electromagnetic wave, where the system's absorption significantly decreases.

4.1.4. Dependence of the absorption coefficient on the intensity of a strong electromagnetic wave E_0

Investigating the dependence of the absorption coefficient on the intensity of a strong electromagnetic wave as the system's temperature changes, we obtained the following results (see Fig. 4).

Figures 4(a) and 4(b) show that the absorption coefficient depends on and is nonlinear with respect to the intensity of electromagnetic waves. As the intensity of strong electromagnetic wave



(a) (b)

Fig. 4. (a) Dependence of the total absorption coefficient on the electromagnetic wave intensity E_0 at different temperatures T. (b) Dependence of the total absorption coefficient on the electromagnetic wave intensity E_0 and the temperature T.

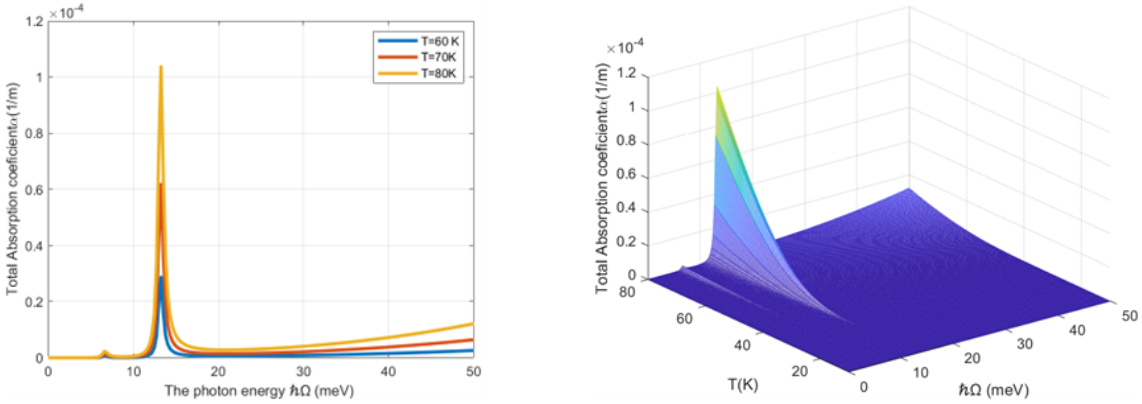
increases, the absorption coefficient also increases nonlinearly. As the electromagnetic wave intensity of the system increases, the absorption coefficient gradually rises, and this increase is more pronounced at higher temperatures ($T = 80$K) compared to lower temperatures ($T = 70$K, 60K). Additionally, higher temperatures enhance the phonon density of states, leading to an increase in the system's absorption process. The electromagnetic wave intensity increases the interaction intensity between electromagnetic waves and electrons, effectively stimulating scattering processes. Moreover, the electromagnetic wave intensity enhances photon absorption, further facilitating electron-phonon scattering. However, this increase is nonlinear, as the absorption coefficient also depends on factors such as wave frequency and temperature.

4.2. Presence of an external magnetic field

4.2.1. Dependence of the absorption coefficient on the photon energy ($\hbar\Omega$) as the system's temperature T changes

By studying the dependence of the absorption coefficient on the photon energy as the system's temperature T changes, we obtained the following results (see Fig. 5).

In Fig. 5a and Fig. 5b, we show the dependence of the absorption coefficient on photon energy. We can observe that as the photon energy increases, resonance peaks appear. In general, these resonance peaks tend to increase as both the photon energy increase. The position of the peaks follows the magneto-photon-phonon resonance condition (MPPRC): $l\hbar\Omega = \varepsilon_{N,n} - \varepsilon_{N',n'} = \hbar\omega_B(N' - N) + 2\hbar\omega_z(n' - n)$. The positions of the resonance peaks calculated according to the resonance condition are obtained as: $\hbar\Omega = 6.59$ meV and $\hbar\Omega = 13.18$ meV. These results are approximately consistent with the positions of the resonance peaks shown on the graph. Based on the MPPRC resonance condition, we can see that temperature does not affect the position of the resonance peaks. Photon absorption occurs when the photon energy $\hbar\Omega$ matches the energy equal



(a) (b)

Fig. 5. Dependence of the total absorption coefficient on the photon energy when the temperature T changes.

to the quantum system. This leads to resonance peaks in the absorption graph. Under the influence of an external magnetic field, the field can alter the quantum states in the ISPPSISQW, altering the distribution of the quantum state density. This affects the position and intensity of the photon absorption peaks, resulting in changes in the location of the maxima on the graph. The large absorption peak at approximately 13.58 meV may be associated with energy transitions between the main quantum states in the quantum well. The smaller peaks appearing at lower energies are the result of secondary interactions between electrons and phonons or due to higher-order excitations.

4.2.2. Dependence of the absorption coefficient on the photon energy ($\hbar\Omega$) as the external magnetic field B changes

By studying the dependence of the absorption coefficient on the photon energy as the external magnetic field B changes, we obtained the following results (see Fig. 6).

In Fig. 6a and Fig. 6b, we examine the dependence of the absorption coefficient on photon energy under the influence of an external magnetic field. Under the influence of the external magnetic field, resonance peaks appear. In general, as the photon energy increases, the resonance peaks increase gradually. According to the MPPRC resonance condition, the positions of the resonance first peaks at $B = 8$ T is $\hbar\Omega = 3.95$ meV, at $B = 9$ T is $\hbar\Omega = 4.61$ meV, and at $B = 10$ T is $\hbar\Omega = 5.27$ meV, respectively. As the magnetic field increases, the resonance peaks gradually shift to the right. The calculated resonance peak positions are approximately consistent with those shown on the graph. Based on the MPPRC resonance condition, we observe that, under the influence of an external magnetic field, the resonance peaks shift to the right.

The external magnetic field quantizes the motion of electrons in the ISPPSISQW, leading to the formation of Landau energy levels. As the magnetic field (B) increases, the energy separation between the Landau levels grows, causing the absorption peaks to shift towards higher photon energies (shift to the right on the graph). A higher magnetic field, such as $B = 10$ T, reduces the absorption intensity at lower energy levels but enhances it at higher energy levels. This can

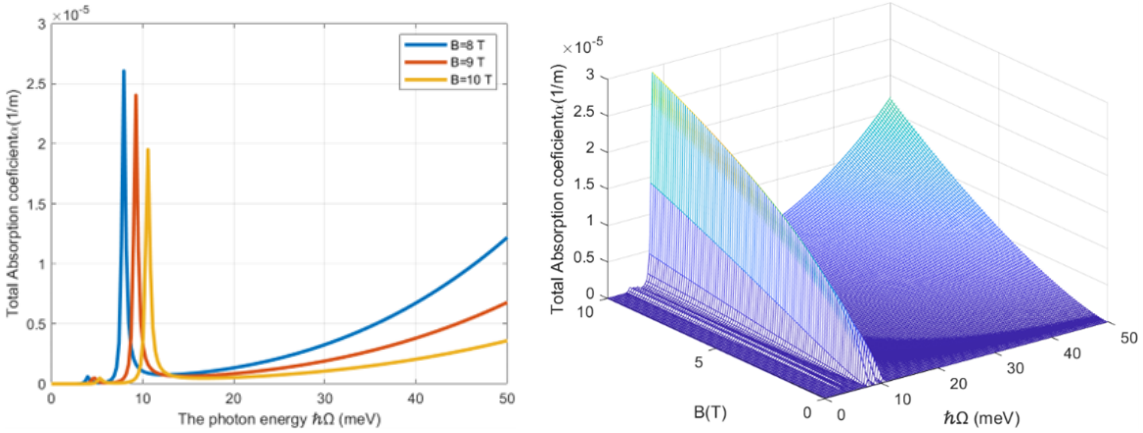


Fig. 6. Dependence of the total absorption coefficient on the photon energy when the external magnetic field B changes.

be attributed to the increased energy gap between Landau levels, which lowers the probability of photon absorption at lower energy levels while simultaneously enhancing the likelihood of multi-photon absorption at higher energy levels.

The expressions for the absorption coefficient (α), the distribution function ($f_{n,\vec{k}_\perp}(t)$), and the total current density ($J_\perp(t)$) are completely different in the case of electron-optical phonon scattering compared to the case of electron-acoustic phonon scattering. For electron-optical phonon scattering, the phonon frequency is high, with a value of $\hbar\omega_0 = 36.25$ meV, and the interaction coefficient $|C(\vec{q})|^2$ does not depend on the wave vector $\vec{q}$. Meanwhile, for electron-acoustic phonon scattering, the acoustic phonon frequency is small and depends on $\vec{q}$, so the expressions contain delta functions, such as in Eq. (11). The acoustic phonon energy is negligible compared to the energy of the electron and the electromagnetic wave, leading to its omission in the delta function. Furthermore, the interaction coefficient for electron-acoustic phonon scattering depends on $\vec{q}$. These two factors result in a completely different transformation from summation to integration, and thus, the outcomes for the two cases are different. This distinction remains true in the presence of an external magnetic field.

In numerical calculations, when comparing with electron-optical phonon scattering in ISPPISQW, we observe that the dependence of the absorption coefficient on the electromagnetic wave frequency exhibits a resonance peak that follows the resonance condition: $\varepsilon_{n'} - \varepsilon_n + \hbar\omega_0 = l\hbar\Omega$. As a result, the numerical results show a resonance peak at 36.5 meV when $\hbar\Omega = \hbar\omega_0$. In contrast, for electron-acoustic phonon scattering, there is no resonance peak; instead, there are nonlinear curves, and the absorption coefficient is very small. Unlike quantum wells with different confinement potentials, ISPPISQW is characterized by a confinement frequency ω_z. We investigated the dependence of the absorption coefficient on ω_z, which revealed a resonance peak at $\omega_z = 0.25 \times 10^{12}$ Hz.

5. Conclusion

In this paper, we are analytically investigating the nonlinear multi-photon Absorption of Strong electromagnetic wave with electron-acoustic phonon scattering by electrons confined in ISPPSISQW by using the quantum kinetic equation. Starting from the Hamiltonian of the electron-acoustic phonon system, we have obtained the analytical expression for the multi-photon nonlinear absorption coefficient in the case without an external magnetic field and in the case with an external magnetic field. The general expression for the absorption coefficient depends on the parameters of strong electromagnetic waves: the strong electromagnetic wave intensity (E_0), the frequency of electromagnetic wave (Ω), external parameters such as temperature T and the magnetic field B, and characteristic parameters of the Quantum Wells such as the confinement frequency of ISPPSISQW ω_z. In the case of magnetic field absence, the numerical results show that as the temperature and the electromagnetic wave intensity of the system increased, the absorption coefficient increased. However, when the frequency of electromagnetic wave of the system and as the confinement frequency increases, the absorption coefficient decreases significantly. When β_z (Parameter characterizing the asymmetric nature of the quantum wells) approaches zero, the current results have the same trend but differ quantitatively from the results of the parabolic symmetric quantum well.

In the case of the presence of an external magnetic field, examining the dependence of the absorption coefficient on photon energy reveals the appearance of resonance peaks. The positions of the resonance peaks follow the MPPRC resonance condition. As the photon energy $(\hbar\Omega)$ changes, the resonance peaks gradually shift to the right with increasing photon energy. When the system's temperature increases, the resonance peaks also increase. This indicates that the material's photon absorption increases with rising temperature (T), likely due to enhanced thermal vibrations and the redistribution of the energy states of particles within the material. Additionally, as the magnetic field changes, leading to a shift to the right of the resonance peaks. This indicates that the magnetic field affects the position of the resonance peaks.

The quantum kinetic equation method we used in this paper can be employed to study the quantum theory of the physical properties of low-dimensional systems, such as quantum wells, cylindrical quantum wires with parabolic potential wells, and two-dimensional graphene. The results obtained in this paper are not applicable to generalized models such as advanced material systems, two-dimensional, and one-dimensional systems. This is explained by the differences in the wave functions and energy spectra of electrons, leading to differences in the general expression of the absorption coefficient.

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Conflict of interest

The authors have no conflict of interest to declare.

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Appendix A. Process of determining the current density

To obtain the results of Eq. (11), we start from the expression of the total current density. First, substitute the result of the electron distribution function (10) into $J_\perp(t)$ and we will proceed to find the result of the expression

$$\frac{e\hbar}{m_e}\sum_{n,\vec{k}_\perp}\vec{k}_\perp f_{n,\vec{k}_\perp}(t). \tag{A.1}$$

Second, with that equation, we can separate it into four components and perform a variable substitution: $\vec{k}_\perp-\vec{q}_\perp=\vec{k}_\perp$, $\vec{k}_\perp=\vec{q}_\perp+\vec{k}_\perp$, $\vec{q}_\perp=-\vec{q}_\perp$, $C(\vec{q}_\perp)=C(-\vec{q}_\perp)$, $N_{\vec{q}_\perp}=N_{\vec{q}_\perp}$ and applying the properties of Bessel functions and considering only the real part,

$$\begin{aligned}
&\frac{e\hbar}{m_e}\sum_{n,\vec{k}_\perp}\vec{k}_\perp f_{n,\vec{k}_\perp}(t)\\
=&\frac{2\pi\hbar e}{m_e k\Omega}\sum_{n,n'}\sum_{\vec{q}_\perp}|C(\vec{q})|^2|I_{n,n'}(\vec{q})|^2\sum_{n,k}\vec{k}_\perp J_l\left(\frac{e\vec{E}_0\vec{q}}{m\Omega}\right)\\
&\times\left[(-1)^k J_{i,k}\left(\frac{e\,frac\vec{c}E_0\vec{q}}{m\Omega}\right)\right]\left(\overline{f_{n,k_\perp+\vec{q}_\perp}}-\overline{f_{n,\vec{k}_\perp}}\right)\left(2N_{\vec{q}+1}\right)\delta\left(\varepsilon_{n',\vec{k}_\perp+\vec{q}_\perp}-\varepsilon_{n,\vec{k}_\perp}-\ell\hbar\Omega\right)\sin\left(k\Omega t\right).
\end{aligned} \tag{A.2}$$

We observe that substituting the expression for the current density $J_\perp(t)$ into the expression for the absorption coefficient with $k=1$ yields meaningful results. Therefore, we consider $k=1$. As a result, we fully obtain Eq. (11).

Appendix B. Process of calculating the absorption coefficient

When calculating the multi-photon nonlinear absorption coefficient in the absence of an external magnetic field, we start from the expression of the Hamiltonian of the electron–phonon system in ISPPSISQW (6) and substitute it into the quantum kinetic equation (9). Our goal is to determine the electron distribution function (10) based on the quantum kinetic equation. First, we compute $\left[a^+_{n,\vec{k}_\perp}a_{n,\vec{k}_\perp},\mathscr{H}\right]$ and obtain the result:

$$\begin{aligned}
\left[a^+_{n,\vec{k}_\perp}a_{n,\vec{k}_\perp},\mathscr{H}\right]=&-\sum_{\vec{q}}C(\vec{q})I_{n,n'}(\vec{q})\left[\left(a^+_{n,\vec{k}_\perp}a_{n',\vec{k}_\perp-\vec{q}_\perp}b_{\vec{q}}\right)+\left(a^+_{n,\vec{k}_\perp}a_{n',\vec{k}_\perp-\vec{q}_\perp}b_{-\vec{q}}\right)^\star\right.\\
&\left.-\left(a^+_{n,\vec{k}_\perp}a_{n',\vec{k}_\perp+\vec{q}_\perp}b_{-\vec{q}}\right)-\left(a^+_{n,\vec{k}_\perp}a_{n',\vec{k}_\perp+\vec{q}_\perp}b_{-\vec{q}}\right)\right]\\
=&\sum_{\vec{q}}C(\vec{q})I_{n,n'}(\vec{q})\left[F_{n',\vec{k}_\perp+\vec{q}_\perp,n,\vec{k}_\perp,\vec{q}}(t)+F_{n,\vec{k}_\perp,n',\vec{k}_\perp+\vec{q}_\perp,-\vec{q}}(t)^\star\right.\\
&\left.-F_{n,\vec{k}_\perp,n',\vec{k}_\perp-\vec{q}_\perp,\vec{q}}(t)-F_{n',\vec{k}_\perp-\vec{q}_\perp,n,\vec{k}_\perp,-\vec{q}}(t)^\star\right].
\end{aligned} \tag{B.1}$$

Secondly, we determine the expression for $F(t)$:

$$\frac{i\partial F(t)}{\partial t}=\langle[F,H]\rangle_t \text{ with } F(t)=\langle a^+_{n,\vec{k}_{1\perp}}a_{n',\vec{k}_{2\perp}}b_{\vec{q}}\rangle_t, \tag{B.2}$$

solving similarly to $\langle a^+_{n,\vec{k}_\perp} a_{n,\vec{k}_\perp}, \mathscr{H}\rangle$, we obtain a differential equation and solve for $F(t)$. Using the result of $F(t)$, we solve equation (9) and obtain equation (10). With the explanation in the first comment, we demonstrated the method to determine the expression for the current density. Substituting equation (11) into the general expression for the absorption coefficient we obtain the total absorption coefficient of the electromagnetic wave in ISPPSISQW in the general form (12). In the paper, we have detailed the transformations and obtained the final result to calculate the nonlinear absorption of strong electromagnetic waves in ISPPSISQW for the case of electron-acoustic phonon scattering (16).

In the presence of an external magnetic field, the electron wavefunction and energy spectrum change. The electron creation and annihilation operators now include the index N (Landau level index), and the Hamiltonian expression (24) introduces the electron form factor (26). However, our approach in this case remains similar to the case without an external magnetic field. The results for the electron distribution function, current density, and absorption coefficient are nearly identical to those in the absence of an external magnetic field. Therefore, we did not include them in the paper. Moreover, due to the changes in the wavefunction and energy spectrum of the system, the integral of the expression D^{γ}_{ℓ} is affected, and we have provided a detailed calculation of this part in the paper.

Appendix C. Calculation of the expression for $n_0^\star$

To calculate the expression of $n_0^\star$, we start from the following expression:

$$N_0 = \sum_i f_{N,n,k_y} = \sum_{in_0^\star} e^{\frac{\varepsilon_{N,n,k_y}}{k_B T}} \tag{C.1}$$

where $N_0 = n_e V_0$ is the total number of electrons in the system, n_0 is the electron density, and $n_0^\star$ is the normalization factor. We then convert the sum into an integral over k_y, and the result for N_0 is given by:

$$N_0 = \frac{L_y}{2\pi} n_0^\star \frac{L_x}{l_B^2} \sum_{N,n} e^{\frac{\varepsilon_{N,n}}{k_B T}} = n_0 V_0, \tag{C.2}$$

From here, we obtain the normalization factor $n_0^\star$ in the presence of a magnetic field:

$$n_0^\star = \frac{2\pi n_0 l_B^2 L_z}{\xi} \text{ with } \xi = \sum_{N,n} e^{-\frac{\varepsilon_{N,n}}{k_B T}}, \tag{C.3}$$

By performing similar calculations in the absence of an external magnetic field, we obtain the result:

$$n_0^\star = \frac{2\pi\hbar^2 n_0}{S m^\star k_B T \xi} \text{ with } \xi = \sum_n e^{-\frac{\varepsilon_n}{k_B T}}, \tag{C.4}$$

Appendix D. Detailed Expressions for Clarification of $G_{n,n'}$

Below is the detailed representation of $G_{n,n'}$:

$$G_{n,n'} = \int_{-\infty}^{+\infty} \left|I_{n,n'}(\vec{q})\right|^2 d_{q_z} = \int_{-\infty}^{+\infty} \left|\langle\varphi_{n'}(z)|e^{iq_z z}|\varphi_n(z)\rangle \delta_{\vec{k}_\perp,\vec{k}_\perp+\vec{q}_\perp}\right|^2 d_{q_z}, \tag{D.1}$$

Using numerical methods, we determine:

$$G_{n,n'} = \frac{2^{-5/2-\sqrt{1+4\beta_z}}\pi(3+4\sqrt{1+4\beta_z})\Gamma(3/2+\sqrt{1+4\beta_z})}{\alpha_z\Gamma[1/2+1/2(1+\sqrt{1+4\beta_z})]\Gamma[3/2+1/2(1+\sqrt{1+4\beta_z})]}. \tag{D.2}$$

For the expression of $(\gamma^{\pm}_{N,N',n,n'})^2$, we used equations (7), (8), and (25). Subsequently, we transformed the summation into an integral over the wave vector $\vec{q}$, and the result of the expression was obtained using numerical methods.